

EXACT SOLUTIONS TO THE COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH THE CORIOLIS AND FRICTION TERMS

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ABSTRACT. We consider special solution to the 3D Navier-Stokes system with and without the Coriolis force and dry friction and find the respective initial data implying a finite time gradient catastrophe. The paper can be considered as extension of the results [1].

We consider the following gas-dynamic like system:

$$(1) \quad \frac{\partial \rho}{\partial t} + (\mathbf{v}, \nabla) \rho + \rho(\nabla, \mathbf{v}) = 0,$$

$$(2) \quad \rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}, \nabla) \mathbf{v} \right) = -\nabla p + \rho(L\mathbf{v} + \mu \Delta \mathbf{v}),$$

$$(3) \quad \frac{\partial p}{\partial t} + (\mathbf{v}, \nabla p) + \gamma p \operatorname{div} \mathbf{v} = 0,$$

where $\rho(t, \mathbf{x})$, $p(t, \mathbf{x})$, $\mathbf{v}(t, \mathbf{x}) = (v_1, v_2, v_3)$ are density, pressure and velocity vector, respectively, μ is the dynamic viscosity coefficient and

$$L = \begin{pmatrix} -m & -l & 0 \\ l & -m & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$m = \text{const} \geq 0$ is the friction coefficient, $l = \text{const}$ is the Coriolis parameter, $\mathbf{x} \in \mathbb{R}^n$, $t \geq 0$.

Exact solutions of the system (1)-(3) has been an area of intensive research activity in the last decades (e.g. [2], [3], [4], [5], [6]). Examples of exact solutions with a special initial distribution of the tangential component are given in [7].

Below we consider the solution to (1)-(3) in several particular cases.

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1. CASE $l = 0$, $m = 0$ (WITHOUT THE CORIOLIS FORCE AND DRY FRICTION)

Let us consider the velocity field and the density in the following form:

$$(4) \quad \mathbf{v} = \begin{pmatrix} \alpha(t)x \\ \beta(t)y \\ W(x, t) \end{pmatrix}, \quad \rho(t, x) = \sigma(t) + \frac{\rho_0}{U}(\alpha(t) + \beta(t))x.$$

Here $\alpha(t) + \beta(t)$ is the divergency of velocity field.

Firstly we are going to define the functions $\rho(x, t)$, $\alpha(t)$ and $\beta(t)$.

For this class of solution the conservation of mass (1) and vorticity conservation equation

$$(5) \quad \nabla \times [\rho(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}, \nabla)\mathbf{v}) - \mu \Delta \mathbf{v}] = 0,$$

which follows from (2), gives

$$(6) \quad \dot{\sigma} + \frac{\rho_0}{U}(\dot{\alpha} + \dot{\beta})x + \alpha \frac{\rho_0}{U}x(\alpha + \beta) + [\sigma + \frac{\rho_0}{U}(\alpha + \beta)x](\alpha + \beta),$$

$$(7) \quad \begin{aligned} & [\sigma + \frac{\rho_0}{U}(\alpha + \beta)x] \left[\frac{\partial^2 W}{\partial x \partial t} + \alpha x \frac{\partial^2 W}{\partial x^2} \right] + \frac{\rho_0}{U}(\alpha + \beta) \left[\frac{\partial W}{\partial t} + \alpha x \frac{\partial W}{\partial x} \right] = \\ & = -\alpha \frac{\partial W}{\partial x} \left[\sigma + \frac{\rho_0}{U}(\alpha + \beta)x \right] + \mu \frac{\partial^3 W}{\partial x^3}, \end{aligned}$$

$$(\alpha + \beta)(\dot{\beta} + \beta^2) = 0.$$

In [1] it was shown that $\alpha(t)$, $\beta(t)$ and $\sigma(t)$ satisfy the following system:

$$(8) \quad \beta(t) = \frac{1}{t+B}, \quad -\alpha(t) = c(t) + \beta(t),$$

$$(9) \quad c(t) = \frac{\exp\left(-\int_0^t \alpha(\tau) d\tau\right)}{C + \int_0^t \exp\left(\int_0^{\tau} \alpha(\tau') d\tau'\right) d\tau},$$

where $\sigma(t) = \exp\left(\int_0^t c(\tau) d\tau\right)$ and the constants B and C can be found from the initial data, namely: $B = (\beta(0))^{-1}$, $C = (c(0))^{-1} = -(\alpha(0) + \beta(0))^{-1}$.

From (8) and (9) we find functions $\alpha(t)$, $\beta(t)$ and $c(t)$. Substituting $-\alpha(t)$ for $c(t) + \frac{1}{t+B}$ in (52), we obtain:

$$(10) \quad c(t) = \frac{(t+B) \exp\left(\int_0^t c(\tau) d\tau\right)}{C - \int_0^t (\tau+B) \exp\left(\int_0^\tau c(\tau') d\tau'\right) d\tau},$$

Let us denote $p(t) = C - \int_0^t (\tau+B) \exp\left(\int_0^\tau c(\tau') d\tau'\right) d\tau$. It is easy to see that

$$c(t) = \left(\ln \left| \frac{p'(t)}{t+B} \right| \right)'.$$

Thus, (10) implies:

$$(11) \quad \frac{p'(t)}{t+B} = \frac{C_1}{p(t)},$$

therefore $p^2(t) = C_1(t^2 + 2Bt + C_2)$. Taking into account (3), we obtain:

$$c(t) = -\frac{p'(t)}{p(t)} = -\frac{C_1(t+B)}{p^2(t)} = -\frac{t+B}{t^2 + 2Bt + C_2},$$

where $C_1 = -\frac{C}{B}$, $C_2 = -BC$.

Therefore, we can find all functions:

$$(12) \quad \alpha(t) = -\frac{1}{t+B} + \frac{t+B}{t^2 + 2Bt + C_2}, \quad \beta(t) = \frac{1}{t+B},$$

$$(13) \quad \sigma(t) = \frac{C_2^{1/2}}{\sqrt{t^2 + 2Bt + C_2}},$$

$$(14) \quad \rho(x, t) = \sigma(t) - \frac{\rho_0}{U} c(t)x = \frac{C_2^{1/2}}{\sqrt{t^2 + 2Bt + C_2}} + \frac{\rho_0}{U} \frac{t+B}{t^2 + 2Bt + C_2} x$$

Having the analytic form of the solution, we can find initial data implying an unbounded increasing of its derivative in a finite time (gradient catastrophe). It is clear that it is sufficient to find when the $f(t) = (t+B)(t^2 + 2Bt - BC)$ has the positive zeroes. We have the following results:

- 1) if $B > 0$, $C \leq 0$ ($\beta(0) > 0$, $\alpha(0) \geq -\beta(0)$) then $f(t) \neq 0$, $\forall t > 0$;
- 2) $B > 0$, $C > 0$ ($\beta(0) > 0$, $\alpha(0) > -\beta(0)$) then $f(t)$ has one positive zero:

$$t = -B + \sqrt{B^2 + BC} = \frac{1}{\beta(0)} \left(\sqrt{\frac{\alpha(0)}{\alpha(0) + \beta(0)}} - 1 \right);$$

- 3) if $B < 0$, $C < 0$ or $C > -B$ ($\beta(0) < 0$, $\alpha(0) > 0$, $\alpha(0) \neq \beta(0)$) then $f(t)$ has one positive zero $t = -B$;

4) if $C = 0$, $B < 0$ ($\beta(0) < 0$, $\alpha(0) = -\beta(0)$), then $f(t) = (t + B)(t^2 + 2Bt)$. Thus, $f(t) = 0$ in the points $t = T_1 = -B$ and $t = T_2 = -2B$;

5) if $B < 0$, $0 < C < -B$ ($\beta(0) < 0$, $\alpha(0) < 0$), then $f(t)$ turns into zero in three points:

$$t = T_1 = -B - \sqrt{B^2 + BC} = \frac{1}{\beta(0)} \left(\sqrt{\frac{\alpha(0)}{\alpha(0) + \beta(0)}} - 1 \right),$$

$$t = T_2 = -B,$$

$$t = T_3 = -B + \sqrt{B^2 + BC} = -\frac{1}{\beta(0)} \left(\sqrt{\frac{\alpha(0)}{\alpha(0) + \beta(0)}} + 1 \right).$$

It is obviously that $T_1 < T_2 < T_3$.

Now we are ready to find the third component of velocity $W(t, x)$.

According to [1] $W(t, x)$ solves the PDE:

$$(15) \quad \rho \left(\frac{\partial W}{\partial t} + \alpha(t)x \frac{\partial W}{\partial x} \right) = \mu \frac{\partial^2 W}{\partial x^2}$$

We consider $W(x, t) = W(\omega(t)x - \lambda(t)) =: W(f)$. Thus, we have:

$$\frac{\partial W}{\partial t} = (\dot{\omega}(t)x - \dot{\lambda}(t))W'(f),$$

$$\frac{\partial W}{\partial x} = \omega(t)W'(f), \quad \frac{\partial^2 W}{\partial x^2} = \omega^2(t)W''(f).$$

Using the specific form of $\rho(x, t)$ (4), we obtain from (15):

$$(16) \quad (\sigma(t) + g(t)x)((\dot{\omega}(t) - a(t)\omega(t))x - \dot{\lambda}(t))W'(f) = \mu\omega^2(t)W''(f).$$

Equation (16) implies:

$$(17) \quad \omega^{-2}(t)(\sigma(t) + g(t)x)((\dot{\omega}(t) - a(t)\omega(t))x - \dot{\lambda}(t)) = (\omega(t)x - \lambda(t))^2$$

Under this condition we find $\omega(t)$ and $\lambda(t)$. From (17) we obtain:

$$(18) \quad g(t)(\dot{\omega}(t) - a(t)\omega(t)) = \omega^4(t),$$

$$(19) \quad -\dot{\lambda}(t)g(t) + \sigma(t)(\dot{\omega}(t) - a(t)\omega(t)) = -2\omega^3(t)\lambda(t),$$

$$(20) \quad -\dot{\lambda}(t)\sigma(t) = \lambda^2(t)\omega^2(t).$$

In this system $\omega(t)$ and $\lambda(t)$ are unknown and $\sigma(t) = \frac{C_2^{1/2}}{\sqrt{t^2 + 2Bt + C_2}}$, $g(t) = \frac{\rho_0}{U} \frac{t + B}{t^2 + 2Bt + C_2}$ (See (13) and (14)).

Using (19) and (20), we find

$$(21) \quad \lambda(t) = -\frac{\sigma(t)}{g(t)}\omega(t) = -\frac{UC_2^{1/2}}{\rho_0} \frac{\sqrt{t^2 + 2Bt + C_2}}{t + B} \omega(t).$$

Let us denote $s(t) = \omega^{-3}(t)$. Thus, from (18) we get

$$\dot{s}(t) + 3a(t)s(t) = -3g^{-1}(t).$$

It can be readily concluded that

$$(22) \quad \omega(t) = \frac{C_2^{1/2}(t + B)}{B\sqrt{t^2 + 2Bt + C_2}} (F(t) + C_3)^{-1/3},$$

where

$$(23) \quad F(t) = -\frac{1}{2}K_1(t + B)\sqrt{t^2 + 2Bt + C_2} + \frac{1}{2}K_1(C_2 - B^2)\ln|t + B + \sqrt{t^2 + 2Bt + C_2}|$$

and

$$K_1 = -3\frac{UC_2^{3/2}}{B^3\rho_0}, \quad C_3 = \omega(0) - F(0).$$

From (21) we find

$$(24) \quad \lambda(t) = K_2 (F(t) + C_3)^{-1/3},$$

where $K_2 = -\frac{UC_2^{1/2}K_1}{\rho_0}$.

Finally we show that (18)-(20) are compatible. Substituting $\omega(t)$ and $\lambda(t)$ (see (22) and (24)) in (20), we obtain

$$(25) \quad \frac{K_2}{3} (F(t) + C_3)^{-4/3} F'(t) \frac{C_2^{1/2}}{q(t)} = K_2^2 (F(t) + C_3)^{-4/3} \frac{C_2(t + B)^2}{B^2 q^2(t)}$$

(here $q(t) = \sqrt{t^2 + 2Bt + C_2}$).

Further, (23) gets

$$F'(t) = -\frac{1}{2}K_1q(t) - \frac{1}{2}K_1 \frac{(t + B)^2}{q(t)} + \frac{1}{2}K_1(C_2 - B^2) \frac{1}{q(t)} = -\frac{K_1(t + B)^2}{q(t)}.$$

Therefore, it is easy to see that (25) holds identically.

For $\omega(t)$ and $\lambda(t)$ defined from (17) we have $\mu W''(f) = f^2 W'(t)$. Integrating gives

$$W(f) = W'(0) \int \exp\left(\frac{1}{3\mu}f^3\right) df,$$

where $f(x, t) = \omega(t)x - \lambda(t)$.

2. CASE $l \neq 0, m \neq 0$

We will find the velocity vector and the density in the following form:

$$(26) \quad \mathbf{v} = \begin{pmatrix} \alpha(t)x + \gamma(t)y \\ \xi(t)x + \beta(t)y \\ W(x, y, t) \end{pmatrix}, \quad \rho(t, x) = \sigma(t) + g(t)x = \sigma(t) + \frac{\rho_0}{U}(\alpha(t) + \beta(t))x.$$

Here the divergency is $\alpha(t) + \beta(t)$, the vorticity is $\left(\frac{\partial W}{\partial y}, -\frac{\partial W}{\partial x}, \xi(t) + \gamma(t)\right)^T$.

Firstly we find the functions $\rho(x, t)$, $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $\xi(t)$.

The conservation of mass (1) yields:

$$\dot{\sigma} + \dot{g}x + (\alpha x + \gamma y)g + (\sigma + gx)(\alpha + \beta) = 0.$$

Therefore

$$(27) \quad \dot{g} + \alpha g + g(\alpha + \beta) = 0,$$

$$(28) \quad \dot{\sigma} + \sigma(\alpha + \beta) = 0,$$

$$(29) \quad g\gamma = 0.$$

From the vorticity conservation equation (7) we have the following equations:

$$(30) \quad \begin{aligned} & \rho \left(\frac{\partial^2 W}{\partial t \partial y} + \gamma \frac{\partial W}{\partial x} + (\alpha x + \gamma y) \frac{\partial^2 W}{\partial x \partial y} + \beta \frac{\partial W}{\partial y} + (\xi x + \beta y) \frac{\partial^2 W}{\partial y^2} \right) = \\ & = \mu \left(\frac{\partial^3 W}{\partial x^2 \partial y} + \frac{\partial^3 W}{\partial y^3} \right), \end{aligned}$$

$$(31) \quad \begin{aligned} & \rho \left(\frac{\partial^2 W}{\partial t \partial x} + \alpha \frac{\partial W}{\partial x} + (\alpha x + \gamma y) \frac{\partial^2 W}{\partial x^2} + \xi \frac{\partial W}{\partial y} + (\xi x + \beta y) \frac{\partial^2 W}{\partial x \partial y} \right) + \\ & + \frac{\partial \rho}{\partial x} \left(\frac{\partial W}{\partial t} + (\alpha x + \gamma y) \frac{\partial W}{\partial x} + (\xi x + \beta y) \frac{\partial W}{\partial y} \right) = -\mu \left(\frac{\partial^3 W}{\partial x^3} + \frac{\partial^3 W}{\partial x \partial y^2} \right), \end{aligned}$$

$$(32) \quad \begin{aligned} & \rho \left(\dot{\xi} - \dot{\gamma} + (\alpha + \beta)(\xi - \gamma) - l(\alpha + \beta) + m(\xi - \gamma) \right) + \\ & + \frac{\partial \rho}{\partial x} \left(\dot{\xi}x + \dot{\beta}y + (\beta + m)(\xi x + \beta y) + (\xi - l)(\alpha x + \gamma y) \right) = 0. \end{aligned}$$

From (32) we get

$$(33) \quad g(2\dot{\xi} - \dot{\gamma} + (\alpha + \beta + m)(2\xi - \gamma) - 2l\alpha - l\beta) = 0,$$

$$(34) \quad g(\dot{\beta} + \beta^2 + m\beta + (\xi - l)\gamma) = 0,$$

$$(35) \quad \sigma(\dot{\xi} - \dot{\gamma} + (\alpha + \beta + m)(\xi - \gamma) - l\alpha - l\beta) = 0.$$

Below we treat particular cases separately.

1. $g(t) \not\equiv 0$, $\sigma(t) \not\equiv 0$. It follows from (29) that $\gamma(t) \equiv 0$. In this case instead of system (33)-(35) we have:

$$(36) \quad 2\dot{\xi} + 2(\alpha + \beta)\xi - 2l\alpha + 2m\xi - l\beta = 0,$$

$$(37) \quad \dot{\beta} + \beta^2 + m\beta = 0,$$

$$(38) \quad \dot{\xi} + (\alpha + \beta)\xi - l\alpha + m\xi - l\beta = 0.$$

From (36) and (38) we get immediately that $\beta(t) \equiv 0$. To find functions $\alpha(t)$, $\xi(t)$ and $\sigma(t)$ we have the system of differential equations obtained from (27), (28) and (36):

$$\dot{\alpha} + 2\alpha^2 = 0,$$

$$\dot{\sigma} + \alpha\sigma = 0,$$

$$\dot{\xi} + (\alpha + m)\xi - l\alpha = 0.$$

Integrating gives

$$(39) \quad \alpha(t) = \frac{1}{2t + K_1},$$

where $K_1 = (\alpha(0))^{-1}$,

$$(40) \quad \sigma(t) = \frac{K_2}{\sqrt{|2t + K_1|}},$$

where $K_2 = \sigma(0)\sqrt{|K_1|} = \sigma(0)\sqrt{|(\alpha(0))^{-1}|}$,

$$(41) \quad \xi(t) = C(t) \frac{e^{-mt}}{\sqrt{|2t + K_1|}},$$

where $C(t) = K_3 + l \int_0^t \frac{e^{m\tau}}{\sqrt{|2\tau + K_1|}} d\tau$.

If $m = 0$, then we can integrate (41):

$$(42) \quad \xi(t) = \frac{K_4}{\sqrt{|2t + K_1|}} + l,$$

where $K_4 = (\xi(0) - l)\sqrt{|K_1|} = (\xi(0) - l)\sqrt{|(\alpha(0))^{-1}|}$.

Thus, for $g(t) \neq 0$, $\sigma(t) \neq 0$ and $m = 0$ we find the following solution for $\rho(x, t)$ and $\mathbf{v}(t, x, y)$:

$$(43) \quad \rho(t, x) = \frac{\sigma(0)}{\alpha(0)\sqrt{|2t + (\alpha(0))^{-1}|}} + \frac{\rho_0}{U} \frac{x}{2t + (\alpha(0))^{-1}},$$

$$(44) \quad \mathbf{v}_1 = \alpha(t)x + \gamma(t)y = \frac{x}{2t + (\alpha(0))^{-1}},$$

$$(45) \quad \mathbf{v}_2 = \xi(t)x + \beta(t)y = \left(l + \frac{(\xi(0) - l)(\alpha(0))^{-1/2}}{\sqrt{|2t + (\alpha(0))^{-1}|}} \right) x.$$

2. $g(t) \neq 0$, $\sigma(t) \equiv 0$. From (29) we get $\gamma(t) \equiv 0$. From (34) we find $\beta(t)$ as follows:

$$(46) \quad \beta(t) = \begin{cases} 0, & \beta(0) = 0; \\ \frac{mC_1}{e^{mt} - C_1}, & \beta(0) \neq 0 \quad m \neq 0; \\ \frac{1}{t + (\beta(0))^{-1}}, & \beta(0) \neq 0 \quad m = 0; \end{cases}$$

where $C_1 = \frac{b(0)}{m + b(0)}$.

It is easy to see from (27)-(29) and (33)-(35) that functions $\alpha(t)$ and $\xi(t)$ solve the following system:

$$(47) \quad \dot{\xi} + (\alpha + \beta)\xi - l\alpha + m\xi - \frac{l}{2}\beta = 0,$$

$$(48) \quad \dot{\alpha} + \dot{\beta} + \alpha(\alpha + \beta) + (\alpha + \beta)^2 = 0.$$

Further, assume $m = 0$. Using (34) and (47)-(48), we obtain

$$(49) \quad \dot{\xi} + (\alpha + \beta)\xi - l\alpha - \frac{l}{2}\beta = 0,$$

$$(50) \quad \dot{\alpha} + 2\alpha^2 + 3\alpha\beta = 0.$$

If $\alpha(0) = 0$ then the solution of (50) is zero identically. The function $\xi(t)$ can be found from (49). Therefore

$$(51) \quad \xi(t) = \frac{lt + C_3}{2(t + C_2)},$$

where

$$(52) \quad C_2 = (\beta(0))^{-1}, \quad C_3 = 2(\beta(0))^{-1}\xi(0).$$

Thus, in this case we obtain

$$(53) \quad \rho(t, x) = \frac{\rho_0 x}{U(t + C_2)},$$

$$(54) \quad \mathbf{v}_1 = 0, \quad \mathbf{v}_2 = \frac{lt + C_3}{2(t + C_2)}x + \frac{1}{t + C_2}y,$$

where the constants C_2, C_3 are determined early (see (52)).

If $\alpha(0) \neq 0$, we denote $A(t) = (\alpha(t))^{-1}$ and we get from (50):

$$\dot{A}(t) - 3\beta(t)A(t) - 2 = 0.$$

Therefore, we can find

$$(55) \quad \alpha(t) = \frac{1}{(t + C_2)(C_4(t + C_2)^2 - 1)},$$

where

$$(56) \quad C_2 = (\beta(0))^{-1}, \quad C_4 = \frac{(\alpha(0))^{-1} + C_2}{C_2^3}.$$

Then we can find from (49)

$$(57) \quad \xi(t) = \frac{C_5}{\sqrt{|C_4(t + C_2)^2 - 1|}} + \frac{l}{2},$$

where

$$(58) \quad C_5 = (\xi(0) - \frac{l}{2})\sqrt{|C_4 C_2^2 - 1|}.$$

Thus, if $g(t) \neq 0$, $\sigma(t) \equiv 0$, $m = 0$ and $\alpha(0) \neq 0$ we have

$$(59) \quad \rho(t, x) = \frac{\rho_0}{U} \frac{C_4(t + C_2)^2}{(t + C_2)(C_4(t + C_2)^2 - 1)}x,$$

$$(60) \quad \mathbf{v}_1 = \frac{x}{(t + C_2)(C_4(t + C_2)^2 - 1)},$$

$$(61) \quad \mathbf{v}_2 = \left(\frac{C_5}{\sqrt{|C_4(t + C_2)^2 - 1|}} + \frac{l}{2} \right) x + \frac{1}{t + C_2} y.$$

where constants C_2, C_4, C_5 are determined in (56) and (58).

2.1. The case $W(x, y, t) = \omega_1(t)x + \omega_2(t)y + \omega_3(t)$. Early we find the first and the second component of velocity vector. In this section we consider the special form of the third component:

$$W(x, y, t) = \omega_1(t)x + \omega_2(t)y + \omega_3(t).$$

Here (30) and (31) imply:

$$(62) \quad (\sigma + gx)(\dot{\omega}_2 + \gamma\omega_1 + \beta\omega_2) = 0,$$

$$(63) \quad g(\dot{\omega}_1x + \dot{\omega}_2y + \dot{\omega}_3 + (\alpha x + \gamma y)\omega_1 + (\xi x + \beta y)\omega_2) + (\sigma + gx)(\dot{\omega}_1 + \alpha\omega_1 + \xi\omega_2) = 0$$

We treat particular cases separately:

1. $g(t) \not\equiv 0, \sigma(t) \not\equiv 0$. From (62) and (63) we obtain:

$$(64) \quad \dot{\omega}_1 + \alpha\omega_1 + \xi\omega_2 = 0,$$

$$(65) \quad \dot{\omega}_2 + \beta\omega_2 + \gamma\omega_1 = 0,$$

$$(66) \quad \dot{\omega}_3 = 0,$$

Early we proved that in the case $g(t) \not\equiv 0, \sigma(t) \not\equiv 0$ we have $\beta(t) \equiv 0$ and $\gamma(t) \equiv 0$ (see (44) and (45)). Thus, from (65) and (66) one gets

$$\omega_i(t) \equiv \omega_i(0), \quad i = 2, 3.$$

Further, we can find from (64)

$$\begin{aligned} \omega_1(t) &= K(t) \frac{1}{\sqrt{|2t + K_1|}}, \\ K(t) &= \frac{l\omega_2(0)}{m} e^{-mt} \int \frac{e^{mt}}{\sqrt{|2t + K_1|}} dt - l\sqrt{|2t + K_1|} + K_5. \end{aligned}$$

where $K_1 = (\alpha(0))^{-1}$.

If $m = 0$, then

$$K(t) = -\omega_2(0)K_4t - \frac{1}{3}l\omega_2(0)|2t + K_1|^{3/2} + K_6,$$

where $K_4 = (\xi(0) - l)\sqrt{|K_1|}$, $K_6 = -(\omega_1(0) + \frac{1}{3}l\omega_2(0)K_1)\sqrt{|K_1|}$.

2. $g(t) \neq 0$, $\sigma(t) \equiv 0$. Then functions $\omega_i(t)$, $i = 1, 2, 3$ solves the system (64)-(66). It is obvious that $\omega_3(t) \equiv \omega_3(0)$. Functions $\omega_1(t)$ and $\omega_2(t)$ we find for case $m = 0$. We have from (29) and (46) $\gamma(t) \equiv 0$, $\beta(t) = \frac{1}{t + (\beta(0))^{-1}}$. Thus, it is easy to see that

$$\omega_2(t) = \frac{\omega_2(0)}{t + C_2},$$

where $C_2 = (\beta(0))^{-1}$.

Further, we can find $\omega_1(t)$. Namely:

a). If $\alpha(0) = 0$, then $\alpha(t) \equiv 0$ and $\xi(t) = \frac{lt + C_3}{2(t + C_2)}$, where $C_3 = 2C_2\xi(0)$, $C_2 = (\beta(0))^{-1}$ (see (51) and (52)). Therefore, we obtain from (64):

$$(67) \quad \omega_1(t) = -\left(\frac{1}{2}l\omega_2(0)t + C_6 \ln |t + C_2| + C_7\right),$$

where $C_6 = \omega_2(0)(\xi(0) - \frac{1}{2})\beta^{-1}(0)$, $C_7 = -C_6 \ln |C_2| - \omega_1(0)$.

b). If $\alpha(0) \neq 0$, then it follows from (55) that

$$(68) \quad -\int_0^t \alpha(\tau) d\tau = \ln \frac{|t + C_2|}{\sqrt{|C_4(t + C_2)^2 - 1|}}.$$

where $C_2 = (\beta(0))^{-1}$, $C_4 = \frac{(\alpha(0))^{-1} + C_2}{C_2^3}$.

Using (57), (58) and (68), we obtain:

$$\omega_1(t) = K(t) \frac{t + C_2}{\sqrt{|C_4(t + C_2)^2 - 1|}},$$

where

$$K(t) = C_8 - \frac{1}{2}C_4 \arcsin \frac{1}{\sqrt{|C_4|(t + C_2)}} - \frac{\sqrt{|C_4(t + C_2)^2 - 1|}}{2(t + C_2)^2},$$

$$C_8 = \sqrt{|C_4C_2^2 - 1|}\beta(0)(\omega_1(0) + \beta(0)) + \frac{1}{2}C_4 \arcsin \frac{\beta(0)}{\sqrt{|C_4|}}.$$

We find all components of the velocity field (26), where $W(x, y, t) = \omega_1(t)x + \omega_2(t)y + \omega_3(t)$. Therefore, we are ready to study the problem of the gradient catastrophe:

1. If $\alpha(0) \neq 0$, $\beta(0) = 0$ and $\gamma(0) = 0$ then

$$\begin{aligned}\rho(t, x) &= \frac{\sigma(0)}{\alpha(0)\sqrt{|2t + K_1|}} + \frac{\rho_0}{U} \frac{x}{2t + K_1}, \\ \mathbf{v}_1 &= \frac{x}{2t + K_1}, \quad \mathbf{v}_2 = \left(l + \frac{K_4}{\sqrt{|2t + K_1|}} \right) x, \\ \mathbf{v}_3 &= \left(\frac{K_6 - \omega_2(0)K_4t}{\sqrt{|2t + K_1|}} - \frac{1}{3}l\omega_2(0)(2t + K_1) \right) x + \omega_2(0)y + \omega_3(0);\end{aligned}$$

It is obvious that for such velocity field the gradient catastrophe takes place in the time $t = -\frac{1}{2\alpha(0)}$ if $\alpha(0) < 0$.

2. If $\sigma(0) = 0$, $\alpha(0) = 0$, $\beta(0) \neq 0$ and $\gamma(0) = 0$ then

$$\begin{aligned}\rho(t, x) &= \frac{\rho_0 x}{U(t + C_2)}, \\ \mathbf{v}_1 &\equiv 0, \quad \mathbf{v}_2 = \frac{lt + C_3}{2(t + C_2)}x + \frac{1}{t + C_2}y, \\ \mathbf{v}_3 &= -\left(\frac{1}{2}l\omega_2(0)t + C_6 \ln|t + C_2| + C_7\right)x + \frac{\omega_2(0)}{t + C_2}y + \omega_3(0);\end{aligned}$$

In this case the necessary condition of gradient catastrophe is $\beta(0) < 0$ and its time is equal $t = -(\beta(0))^{-1}$.

3. If $\sigma(0) = 0$, $\alpha(0) \neq 0$, $\beta(0) \neq 0$ and $\gamma(0) = 0$ then

$$\begin{aligned}\rho(t, x) &= \frac{\rho_0}{U} \frac{C_4(t + C_2)^2}{(t + C_2)(C_4(t + C_2)^2 - 1)}x, \\ \mathbf{v}_1 &= \frac{x}{(t + C_2)(C_4(t + C_2)^2 - 1)}, \\ \mathbf{v}_2 &= \left(\frac{C_5}{\sqrt{|C_4(t + C_2)^2 - 1|}} + \frac{l}{2} \right) x + \frac{1}{t + C_2}y, \\ \mathbf{v}_3 &= (C_8(t + C_2) - \frac{C_4(t + C_2)}{2\sqrt{|C_4(t + C_2)^2 - 1|}} \arcsin \frac{1}{\sqrt{|C_4|}(t + C_2)} - \frac{1}{t + C_2})x + \\ &\quad + \frac{\omega_2(0)}{t + C_2}y + \omega_3(0).\end{aligned}$$

To solve the problem of the gradient catastrophe we find the initial values of velocity such as the function $f(t) = (C_4(t + C_2)^2 - 1)(t + C_2)$ has positive zeroes. Zeroes of this functions are $t = T_1 = -C_2$ and $t = T_{2,3} = \pm \frac{1}{\sqrt{C_4}} - C_2$. $T_2 > 0$ and $T_3 > 0$ if:

$$\begin{cases} \frac{\beta(0)}{\alpha(0)} + 1 > 0, \\ \frac{(\beta(0))^{-3/2}}{\sqrt{(\alpha(0))^{-1} + (\beta(0))^{-1}}} > \frac{1}{\beta(0)}. \end{cases}$$

Thus, we obtain:

- 1). If $\beta(0) > 0$ and $\alpha(0) > -\beta(0)$ then gradient catastrophe doesn't appear;
- 2). If $\beta(0) > 0$ and $\alpha(0) < -\beta(0)$ then gradient catastrophe appears in the time

$$t = \frac{1}{\beta(0)} \left(\sqrt{\frac{\alpha(0)}{\alpha(0) + \beta(0)}} - 1 \right);$$

- 3). If $\beta(0) < 0$ and $0 < \alpha(0) < -\beta(0)$ then gradient catastrophe appears in the time $T = -(\beta(0))^{-1}$;

- 4). If $\beta(0) < 0$ and $\alpha(0) > -\beta(0)$ then gradient of velocity turns in infinity in two points:

$$t = T_1 = \frac{1}{\beta(0)} \left(\sqrt{\frac{\alpha(0)}{\alpha(0) + \beta(0)}} - 1 \right)$$

$$t = T_2 = -(\beta(0))^{-1}$$

$$(T_1 < T_2);$$

- 5). If $\beta(0) < 0$ $\alpha(0) < 0$ then gradient of velocity turns in infinity in three points:

$$t = T_1 = \frac{1}{\beta(0)} \left(\sqrt{\frac{\alpha(0)}{\alpha(0) + \beta(0)}} - 1 \right),$$

$$t = T_2 = -(\beta(0))^{-1},$$

$$t = T_3 = -\frac{1}{\beta(0)} \left(\sqrt{\frac{\alpha(0)}{\alpha(0) + \beta(0)}} + 1 \right).$$

$$(T_1 < T_2 < T_3).$$

2.2. The case $W(t, x, y) = \omega_1(t)x^2 + \omega_2(t)xy + \omega_3(t)y^2 + \lambda_1(t)x + \lambda_2(t)y + \lambda_3(t)$.

In this section the third component of velocity vector is function $W(t, x, y) = \omega_1(t)x^2 + \omega_2(t)xy + \omega_3(t)y^2 + \lambda_1(t)x + \lambda_2(t)y + \lambda_3(t)$. We get from (30) and (31):

$$(69) \quad g(\dot{\omega}_1 + 2\alpha\omega_1 + \xi\omega_2) = 0,$$

$$(70) \quad g(\dot{\omega}_2 + (\alpha + \beta)\omega_2 + 2\gamma\omega_1 + 2\xi\omega_3) = 0,$$

$$(71) \quad (\sigma + g)(\dot{\omega}_3 + 2\beta\omega_3 + \gamma\omega_2) = 0,$$

$$(72) \quad (2g + \sigma)(\dot{\lambda}_1 + \alpha\lambda_1 + \xi\lambda_2) = 0,$$

$$(73) \quad \sigma(\dot{\omega}_2 + \alpha + \beta\omega_2 + 2\gamma\omega_1 + 2\xi\omega_3) + (g + \sigma)(\dot{\lambda}_2 + \beta\lambda_2 + \gamma\lambda_1) = 0,$$

$$(74) \quad g\dot{\lambda}_3 + \sigma(\dot{\lambda}_1 + \alpha\lambda_1 + \xi\lambda_2) = 0.$$

To solve this system consider several cases:

1. $g(t) \not\equiv 0$, $\sigma(t) \not\equiv 0$. Then we get from (44) and (45) $\beta(t) \equiv 0$ and $\gamma(t) \equiv 0$.

Therefore, we solve the following system:

$$(75) \quad \dot{\omega}_1 + 2\alpha\omega_1 + \xi\omega_2 = 0,$$

$$(76) \quad \dot{\lambda}_1 + \alpha\lambda_1 + \xi\lambda_2 = 0,$$

$$(77) \quad \dot{\omega}_2 + \alpha\omega_2 + 2\xi\omega_3 = 0,$$

$$(78) \quad \dot{\omega}_3 = 0,$$

$$(79) \quad \dot{\lambda}_i = 0, \quad i = 2, 3.$$

It easily follows from (78)-(79) that $\omega_3(t) \equiv \omega_3(0)$, $\lambda_i(t) \equiv \lambda_i(0)$, $i = 2, 3$.

Further, we find from (76) and (77):

$$\omega_2(t) = \frac{2\lambda_2(0)}{\omega_3(0)}\lambda_1(t).$$

We have from (39)

$$(80) \quad \int \alpha(\tau) d\tau = \ln \sqrt{|2t + K_1|}.$$

Using (42) and (80), we can solve (75) and (76):

$$\lambda_1(t) = -\lambda_2(0) \left(\frac{l}{3}(2t + K_1) + \frac{K_4t + K_7}{\sqrt{|2t + K_1|}} \right),$$

$$\begin{aligned}\omega_1(t) = & \frac{\lambda_2(0)l}{18}(2t + K_1)^2 + \frac{lK_7}{3}\sqrt{|2t + K_1|} + \frac{2\lambda_2(0)lK_4}{15}|2t + K_1|^{3/2} + \\ & + \frac{\lambda_2(0)K_4t^2 + 2K_4K_7t}{2(2t + K_1)} + \frac{\lambda_2(0)lK_4}{3}t\sqrt{|2t + K_1|} + \frac{K_8}{2t + K_1},\end{aligned}$$

where

$$K_7 = -\sqrt{|K_1|} \left(\frac{\lambda_1(0)}{\lambda_2(0)} + \frac{lK_1}{3} \right),$$

$$K_8 = \omega_1(0)K_1 - \frac{1}{18}\lambda_2(0)lK_1^3 - \frac{1}{3}lK_7|K_1|^{3/2} - \frac{2}{15}\lambda_2(0)lK_4|K_1|^{5/2}.$$

2. $g(t) \not\equiv 0$, $\sigma(t) \equiv 0$. We get from (29) and (46) $\gamma(t) \equiv 0$ and $\beta(t) = \frac{1}{t + (\beta(0))^{-1}}$. Thus, we obtain the system:

$$(81) \quad \dot{\omega}_1 + 2\alpha\omega_1 + \xi\omega_2 = 0,$$

$$(82) \quad \dot{\lambda}_1 + \alpha\lambda_1 + \xi\lambda_2 = 0,$$

$$(83) \quad \dot{\omega}_2 + (\alpha + \beta)\omega_2 + 2\xi\omega_3 = 0,$$

$$(84) \quad \dot{\lambda}_2 + \beta\lambda_2 = 0,$$

$$(85) \quad \dot{\omega}_3 + 2\beta\omega_3 = 0,$$

$$(86) \quad \dot{\lambda}_3 = 0.$$

Firstly, we get from (86) that $\lambda_3(t) \equiv \lambda_3(0)$.

We have from (46)

$$\int_0^t \beta(\tau) d\tau = \ln|t + C_2|.$$

Therefore, it can be easily calculated that

$$(87) \quad \lambda_2(t) = \lambda_2(0) \frac{1}{t + C_2},$$

$$(88) \quad \omega_3(t) = \omega_3(0) \frac{1}{(t + C_2)^2}.$$

Further, we have two cases:

a). If $\alpha(0) = 0$ then from (50) $\alpha(t) \equiv 0$. For this reason we have the following system:

$$(89) \quad \dot{\lambda}_1 + \xi \lambda_2 = 0,$$

$$(90) \quad \dot{\omega}_1 + \xi \omega_2 = 0,$$

$$(91) \quad \dot{\omega}_2 + \beta \omega_2 + 2\xi \omega_3 = 0,$$

where $\xi(t) = \frac{lt + C_3}{2(t + C_2)}$ (see (51)).

Let us denote

$$(92) \quad I(t) = \int_0^t \frac{l\tau + C_3}{2(\tau + C_2)^2} d\tau = l \ln |t + C_2| + \frac{lC_2 - C_3}{t + C_2},$$

where $C_2 = (\beta(0))^{-1}$, $C_3 = 2(\beta(0))^{-1}\xi(0)$ (see (52)).

Using (92), find the solution of (89) and (91):

$$(93) \quad \lambda_1(t) = -\frac{\lambda_2(0)}{2}(I(t) - C_9),$$

$$(94) \quad \omega_2(t) = -\omega_3(0)(I(t) - C_{10}),$$

where constants C_9 and C_{10} are determined from the initial data, namely:

$$C_9 = \frac{2\lambda_1(0)}{\lambda_2(0)} + l \ln |C_2| + l - \frac{C_3}{C_2},$$

$$C_{10} = \frac{\omega_2(0)}{\omega_3(0)} + l \ln |C_2| + l - \frac{C_3}{C_2}.$$

Finally, we solve (90):

$$(95) \quad \omega_1(t) = \frac{\omega_3(0)}{2}\Omega_1^0(t),$$

where

$$\Omega_1^0(u) = \frac{1}{2}C_2l(1-l)\ln^2|u| + l^2u\ln|u| + C_{11}\ln|u| + (lC_{10} - l^2)u + \frac{C_{12}}{u} + C_{13},$$

(here $u = t + C_2$).

$$C_{11} = C_2 C_{10}(1 - l) + l(lC_2 - C_3),$$

$$C_{12} = C_2(l - 1)(lC_2 - C_3),$$

$$C_{13} = \frac{2\omega_1(0)}{\omega_3(0)} - \frac{1}{2}C_2 l(1 - l) \ln^2 |C_2| - (l^2 C_2 + C_{11}) \ln |C_2| - (lC_{10} - l^2)C_2 - \frac{C_{12}}{C_2}.$$

b). $\alpha(0) \neq 0$, then $\alpha(t) = ((t + C_2)(C_4(t + C_2)^2 - 1))^{-1}$ (see (55)) and (56). We have:

$$-\int_0^t \alpha(\tau) d\tau = \ln \frac{t + C_2}{\sqrt{|C_4(t + C_2)^2 - 1|}}.$$

$\lambda_1(t)$, $\omega_1(t)$ and $\omega_2(t)$ solve the system (81)-(83).

The equation (81) is the same as (62). Therefore, we find:

$$\lambda_1(t) = L(t) \frac{t + C_2}{\sqrt{|C_4(t + C_2)^2 - 1|}} e^t,$$

where

$$L(t) = C_{14} - \frac{1}{2}C_4 \arcsin \frac{1}{\sqrt{|C_4|(t + C_2)}} - \frac{\sqrt{|C_4(t + C_2)^2 - 1|}}{2(t + C_2)^2},$$

$$C_{14} = \frac{\sqrt{|C_4 C_2^2 - 1|}}{C_2} (2C_2 \lambda_1(0) - 1) + \frac{1}{2}C_4 \arcsin \frac{1}{\sqrt{|C_4|C_2}}.$$

We find from (83):

$$\omega_2(t) = \Omega_2(t) \frac{1}{\sqrt{|C_4(t + C_2)^2 - 1|}},$$

where the function $\Omega_2(t)$ can be found from:

$$(96) \quad \Omega_2'(t) = -(l\sqrt{|C_4(t + C_2)^2 - 1|} + 2C_5) \frac{\omega_3(0)}{(t + C_2)^2}.$$

Let us denote $u = t + C_2$, $p(u) = \sqrt{C_4 u^2 - 1}$. Further, we integrate (96). Thus, we obtain:

$$\Omega_2(u) = \omega_3(0) \left(\frac{2C_5}{u} + \frac{l\sqrt{|C_4 u^2 - 1|}}{u} - l\sqrt{|C_4|} \ln |\sqrt{C_4}u + \sqrt{|C_4 u^2 - 1|}| + C_{15} \right),$$

where

$$C_{15} = \frac{\omega_2(0)}{\omega_3(0)} \sqrt{|C_4 C_2^2 - 1|} - \frac{2C_5 + l\sqrt{|C_4 C_2^2 - 1|}}{C_2} + l\sqrt{|C_4|} \ln |2\sqrt{|C_4|}C_2|.$$

Finally, we have from (82)

$$\omega_1(t) = W_1(t) \frac{(t + C_2)^2}{(C_4(t + C_2)^2 - 1)},$$

where $W_1(t)$ can be found from

$$W_1'(t) = -(l\sqrt{|C_4(t + C_2)^2 - 1|} + 2C_5) \frac{\Omega_2(t)}{(t + C_2)^2}.$$

It can be easily calculated that

$$\begin{aligned} W_1(u) = C_4 \left(p(u) + u + \arctan \frac{1}{p(u)} + \ln |u| \right) + \sqrt{C_4} \ln |\sqrt{C_4}u + p(u)| + \\ + \frac{1 - p(u)}{u} + \frac{1}{2u^2} + J(u), \end{aligned}$$

where

$$J(u) = \int \frac{p(u)(p(u) + 1)}{u^2} \ln |\sqrt{C_4}u + p(u)| du.$$

Let us remark that conditions of the gradient catastrophe are the same as for linear function $W(t, x, y)$.

2.3. The case $W(t, x, y) = \omega_1(t)x + \omega_2(t)y + \omega_3(t)z + \omega_4(t)$. We will find the velocity vector and the density

$$(97) \quad \mathbf{v} = \begin{pmatrix} \alpha(t)x + \gamma(t)y \\ \xi(t)x + \beta(t)y \\ W(t, x, y, z) \end{pmatrix}, \quad \rho(t, x) = \sigma(t) + g(t)x = \sigma(t) + \frac{\rho_0}{U}(\alpha(t) + \beta(t))x.$$

Here the divergency is

$$\alpha(t) + \beta(t) + \frac{\partial W}{\partial z},$$

the vorticity is

$$\left(\frac{\partial W}{\partial y}, -\frac{\partial W}{\partial x}, \xi(t) + \gamma(t) \right)^T.$$

As early, we write the conservation of mass and the vorticity conservation equation for our class of solution. Equation (1) gives

$$\dot{\sigma} + \dot{g}x + (\alpha x + \gamma y)g + (\sigma + gx)(\alpha + \beta + \frac{\partial W}{\partial z}) = 0.$$

Therefore

$$(98) \quad \dot{g} + \alpha g + g(\alpha + \beta + \frac{\partial W}{\partial z}) = 0,$$

$$(99) \quad \dot{\sigma} + \sigma(\alpha + \beta + \frac{\partial W}{\partial z}) = 0,$$

$$(100) \quad g\gamma = 0.$$

From (7) we have the following equations:

$$(101) \quad \rho \left(\frac{\partial^2 W}{\partial t \partial y} + \gamma \frac{\partial W}{\partial x} + (\alpha x + \gamma y) \frac{\partial^2 W}{\partial x \partial y} + \beta \frac{\partial W}{\partial y} + (\xi x + \beta y) \frac{\partial^2 W}{\partial y^2} \right) + \frac{\partial W}{\partial y} \frac{\partial W}{\partial z} + W \frac{\partial^2 W}{\partial y \partial z} = \mu \left(\frac{\partial^3 W}{\partial x^2 \partial y} + \frac{\partial^3 W}{\partial y^3} + \frac{\partial^3 W}{\partial y \partial z^2} \right),$$

$$(102) \quad \rho \left(\frac{\partial^2 W}{\partial t \partial x} + \alpha \frac{\partial W}{\partial x} + (\alpha x + \gamma y) \frac{\partial^2 W}{\partial x^2} + \xi \frac{\partial W}{\partial y} + (\xi x + \beta y) \frac{\partial^2 W}{\partial x \partial y} + W \frac{\partial^2 W}{\partial x \partial z} \right) + \frac{\partial \rho}{\partial x} \left(\frac{\partial W}{\partial t} + (\alpha x + \gamma y) \frac{\partial W}{\partial x} + (\xi x + \beta y) \frac{\partial W}{\partial y} + W \frac{\partial W}{\partial z} \right) = -\mu \left(\frac{\partial^3 W}{\partial x^3} + \frac{\partial^3 W}{\partial x \partial y^2} + \frac{\partial^3 W}{\partial x \partial z^2} \right),$$

$$(103) \quad \rho \left(\dot{\xi} - \dot{\gamma} + (\alpha + \beta)(\xi - \gamma) - l(\alpha + \beta) + m(\xi - \gamma) \right) + \frac{\partial \rho}{\partial x} \left(\dot{\xi} x + \dot{\beta} y + (\beta + m)(\xi x + \beta y) + (\xi - l)(\alpha x + \gamma y) \right) = 0.$$

We consider the case $W(t, x, y) = \omega_1(t)x + \omega_2(t)y + \omega_3(t)z + \omega_4(t)$. For such function (101) and (102) imply:

$$(104) \quad (\sigma + gx)(\dot{\omega}_2 + \gamma\omega_1 + \beta\omega_2 + \omega_2\omega_3) = 0,$$

$$(105) \quad g(\dot{\omega}_1 x + \dot{\omega}_2 y + \dot{\omega}_3 + (\alpha x + \gamma y)\omega_1 + (\xi x + \beta y)\omega_2 + \omega_3(\omega_1(t)x + \omega_2(t)y + \omega_3(t)z + \omega_4(t))) + (\sigma + gx)(\dot{\omega}_1 + \alpha\omega_1 + \xi\omega_2 + \omega_1\omega_3) = 0$$

(103) is the same as (32). Therefore, from (103) we obtain the system (33)-(35).

Further, we find the functions $\sigma(t)$, $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\xi(t)$, $\omega_1(t)$, $\omega_2(t)$, $\omega_3(t)$ and $\omega_4(t)$.

We can integrate the system (104)-(105) in the case $g(t) \neq 0$, $\sigma(t) \neq 0$. We have from (100) $\gamma \equiv 0$. Then we may conclude from (33) and (35) that $\beta \equiv 0$.

Thus, we have the system, which follows from (104)-(105) and facts $\gamma \equiv 0$, $\beta \equiv 0$:

$$(106) \quad \dot{\omega}_1 + (\alpha + \omega_3)\omega_1 + \xi\omega_2 = 0,$$

$$(107) \quad \dot{\omega}_2 + \omega_3\omega_2 = 0,$$

$$(108) \quad \dot{\omega}_3 + \omega_3^2 = 0,$$

$$(109) \quad \dot{\omega}_4 + \omega_3\omega_4 = 0,$$

We solve the system (107)-(109) and obtain:

$$(110) \quad \omega_3(t) = \frac{1}{t + c_3},$$

where $c_3 = (\omega_3(0))^{-1}$, and

$$(111) \quad \omega_i(t) = \frac{c_i}{t + c_3},$$

where $c_i = \omega_i(0)c_3$, $i = 2, 4$. Here

$$g(t) = \frac{\rho_0}{U}(\alpha(t) + \beta(t))$$

(see (97)). Therefore, we get from (98)-(99):

$$(112) \quad \dot{\alpha} + 2\alpha^2 + \omega_3\alpha = 0.$$

$$(113) \quad \dot{\sigma} + \sigma(\alpha + \omega_3) = 0,$$

Let us denote $A(t) = (\alpha(t))^{-1}$ and obtain from (112)

$$\dot{A} - \omega_3 A - 2 = 0.$$

Now it can be calculated that:

$$(114) \quad \alpha(t) = (A(t))^{-1} = \frac{t + c_3}{t^2 + 2c_3t + c_5},$$

where $c_5 = (\alpha(0))^{-1}c_3$.

We have from (114):

$$(115) \quad \int_0^t \alpha(\tau) d\tau = \frac{1}{2} \ln |t^2 + 2c_3t + c_5|.$$

Using (115), we solve (113):

$$(116) \quad \sigma(t) = \frac{c_6}{(t + c_3)\sqrt{t^2 + 2c_3t + c_5}},$$

where $c_6 = \sigma(0)c_3\sqrt{c_5}$.

Further, we get from (35)

$$\dot{\xi} + \alpha\xi - l\alpha = 0.$$

Consequently,

$$(117) \quad \xi(t) = l + \frac{c_7}{\sqrt{t^2 + 2c_3t + c_5}},$$

where $c_7 = (\xi(0) - l)\sqrt{c_5}$.

Finally, we find $\omega_1(t)$ from (106)

$$(118) \quad \omega_1(t) = \frac{K(t)}{(t + c_3)\sqrt{t^2 + 2c_3t + c_5}},$$

where

$$K(t) = -c_2c_7t - \frac{1}{2}c_2l((t + c_3)q(t) + (c_5 - c_3^2)\ln|t + c_3 + q(t)|) + c_1,$$

(here $q(t) = \sqrt{t^2 + 2c_3t + c_5}$),

$$c_1 = \omega_1(0)c_3\sqrt{c_5} + \frac{1}{2}c_2l(c_3\sqrt{c_5} + (c_5 - c_3^2)\ln|c_3 + \sqrt{c_5}|).$$

Thus, in this case we obtain:

$$(119) \quad \rho(t, x) = \frac{c_6}{(t + c_3)\sqrt{t^2 + 2c_3t + c_5}} + \frac{\rho_0(t + c_3)}{U(t^2 + 2c_3t + c_5)}x,$$

$$(120) \quad \mathbf{v}_1 = \frac{t + c_3}{t^2 + 2c_3t + c_5}x,$$

$$(121) \quad \mathbf{v}_2 = \left(l + \frac{c_7}{\sqrt{t^2 + 2c_3t + c_5}} \right) x,$$

$$(122) \quad \mathbf{v}_3 = \frac{K(t)}{(t + c_3)\sqrt{t^2 + 2c_3t + c_5}}x + \frac{c_2}{t + c_3}y + \frac{1}{t + c_3}z + \frac{c_4}{t + c_3},$$

where the function $K(t)$ and the constants c_i , $i = 2, \dots, 7$ are determined early.

To study whether the gradient catastrophe arises, we find the positive zeroes of $f(t) = (t + c_3)(t^2 + 2c_3t + c_5)$. Let us denote $B = c_3$, $-BC = c_5$. Thus we obtain the problem just considered in, Section 1.

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